Generalized Harmonic Analysis of Computed and Measured Magnetic Fields

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In this paper we present a generalized approach for the harmonic analysis of the magnetic field in accelerator magnets. This analysis is based on the covariant components of the computed or measured magnetic flux density. The multipole coefficients obtained in this way can be used for magnet optimization and field reconstruction in the interior of circular and elliptical boundaries in the aperture of straight magnets.

***Index Terms*—Accelerator magnets, Magnetic analysis, Fourier series, Laplace equations**

I. Introduction

THE FIELD quality in magnet apertures, or any domain that is free of currents and magnetized material, is conveniently described by a set of Fourier coefficients of the field solution, subsequently denoted field harmonics. One of the objectives in magnet design optimization, using numerical field computation tools, is to suppress the unwanted harmonics as much as possible [1]. The field harmonics are determined from a measured flux linkage or a computed magnetic flux density on the boundary of the domain of interest, which is chosen as the coordinate line of a suitable coordinate system. These coefficients give a series representation of the Dirichlet data on the boundary, from which the field in the entire domain of interest can be restored by harmonic extension. Technically, this is accomplished by comparing coefficients with a general solution of the Laplace equation, which has been obtained by separation of variables.

The goal of this paper is to show that the underlying methodology is the same in all two-dimensional cases where the coordinate system is obtained by conformal mapping from Cartesian coordinates, and provided that a sufficiently general approach is adopted that allows to deal with the different representations of the metric. We briefly recall the situation in polar coordinates because they are the most commonly used for the computation of fields in long accelerator magnets. We show a generalization of the method to elliptic coordinates, which are more suited to magnets where the beam pipe is not circular, e.g., insertion devices for synchrotron light sources. This facilitates the optimization process in numerical field computation as well as its verification by magnetic field measurements.

II. Circular Field Harmonics

A general solution that satisfies the Laplace equation, $\Delta A_z = 0$, can be found by the separation of variables method. As the vector potential is single-valued, it must be a periodic function in φ. Let us further consider the magnet aperture as the problem domain and incorporate the condition that the flux density is finite at $r = 0$. The general solution for the vector potential can then be written as $A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n(\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi)$. The field components can then be expressed as a function of the coefficients $\mathcal{A}_n$ and $\mathcal{B}_n$, which yields in case of the radial component $B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi)$, in Ω_{a}. Notice the appearance of the metric coefficient $1/r$ in the derivative of the vector potential. Each value of the integer n in the solution of the Laplace equation corresponds to a specific flux distribution generated by ideal magnet geometries. Assuming that the radial component of the magnetic flux density is measured or calculated at a reference radius $r = r_0$ as a function of the angular position φ, we obtain the Fourier series expansion of the radial field component, that is, $B_r(r_0,\varphi) = \sum_{n=1}^{\infty}\big(B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi$. Comparing the coefficients in the two expressions for the B_r components we obtain $\mathcal{A}_n = (1/n\, r_0^{n-1})A_n(r_0)$ and $\mathcal{B}_n = (-1/n\, r_0^{n-1})B_n(r_0)$. Thus the radial field component in the entire domain Ω_{a} can be expressed as $B_r(r,\varphi) = \sum_{n=1}^{\infty}(r/r_0)^{n-1}\big(B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi\big)$. The normal and skew multipole coefficients $B_n(r_0)$, $A_n(r_0)$ are given in units of Tesla at a reference radius r_0, usually chosen to about 2/3 of the magnet aperture [1]. In numerical field computation, it is useful to perform a Fourier analysis of the vector potential on the reference radius, thus avoiding the calculation of the flux density by means of numerical differentiation. Fourier series expansion of the magnetic vector potential at a reference radius r_0 yields $A_z(r_0,\varphi) = \sum_{n=1}^{\infty}\big(C_n(r_0)\sin n\varphi + D_n(r_0)\cos n\varphi\big)$. Comparing the coefficients with the earlier expressions we obtain $B_n(r_0) = -(n/r_0)D_n(r_0)$ and $A_n(r_0) = (n/r_0)C_n(r_0)$. The curl operator acting on vector fields has thus been replaced by scaling laws of the corresponding Fourier series. Similar expressions can be derived for the magnetic scalar potential ϕ_{m}.

In general, any 2π-periodic signal can be transformed, using the discrete Fourier transform, into an expression $\sum_{n=1}^{\infty}(C_n(r_0)\sin n\varphi + D_n(r_0)\cos n\varphi)$. Table I relates the Fourier coefficients C_n, D_n, of the signals $B_r, B_\varphi, A_z, \phi_{\mathrm{m}}, B_x, B_y$ to the multipoles B_n, A_n according to the definition above. This technique cannot be applied

TABLE I
RELATIONS BETWEEN THE MULTIPOLE COEFFICIENTS AND THE FOURIER COEFFICIENTS OF THE EXPANSION OF B_r, B_φ, B_x, B_y, AND A_z, ϕ_{m}.

	B_r	B_φ	B_x	B_y	A_z	ϕ_{m}
$B_n =$	C_n	D_n	C_{n-1}	D_{n-1}	$\frac{-nD_n}{r_0}$	$\frac{-n\mu_0 C_n}{r_0}$
$A_n =$	D_n	$-C_n$	D_{n-1}	$-C_{n-1}$	$\frac{nC_n}{r_0}$	$\frac{-n\mu_0 D_n}{r_0}$

directly to elliptical domains, because the metric coefficients depend on both coordinates and the field harmonics cannot be computed by a simple comparison of coefficients. A method to circumvent this problem is presented in the next section.

III. COVARIANT COMPONENTS IN PLANE ELLIPTIC COORDINATES

Consider a plane ellipse, centered at the origin, with major-semi axis a and minor-semi axis b. A system of elliptic coordinates is defined by the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2 : (\eta, \psi) \mapsto (x, y)$ according to $x = e \cosh \eta \cos \psi$, and $y = e \sinh \eta \sin \psi$, where $e = \sqrt{a^2 - b^2}$ is the distance between the origin and the focal points. For the reference ellipse with semi axes $b = e \sinh \eta_0$ and $a = e \cosh \eta_0$ it follows that $\eta_0 = \operatorname{artanh}(b/a)$ for $a > b$. The metric coefficients for the elliptic coordinates are $h_1^2 = h_2^2 = e^2(\cosh^2 \eta - \cos^2 \psi)$. Written in plane elliptic coordinates, the Laplace equation for the vector potential takes the form: $\Delta A_z = (e^2(\cosh^2 \eta - \cos^2 \psi))^{-1} \left(\frac{\partial^2 A_z}{\partial \eta^2} + \frac{\partial^2 A_z}{\partial \psi^2} \right) = 0$. It is shown in [3] that a complete system of orthogonal eigenfunctions is given by $\cos n\psi \cosh n\eta$ and $\sin n\psi \sinh n\eta$. The general solution can therefore be written as $A_z(\eta, \psi) = \sum_{n=1}^{\infty} (\mathcal{A}_n \sinh n\eta \sin n\psi + \mathcal{B}_n \cosh n\eta \cos n\psi)$. The components of the magnetic flux density are calculated from the vector potential by $B_\eta = \frac{1}{h_2} \frac{\partial A_z}{\partial \psi}$ and $B_\psi = -\frac{1}{h_1} \frac{\partial A_z}{\partial \eta}$. Substituting these expressions into the above equation we obtain: $B_\eta(\eta, \psi) = \frac{1}{h_2} \sum_{n=1}^{\infty} (n \mathcal{A}_n \sinh n\eta \cos n\psi - n \mathcal{B}_n \cosh n\eta \sin n\psi)$.

The field component B_η can be obtained from the (numerically calculated or measured) Cartesian components by $B_\eta = 1/h_1 (e \sinh \eta \cos \psi B_x + e \cosh \eta \sin \psi B_y)$. At the reference ellipse $\eta = \eta_0$, we can formally obtain the Fourier series expansion $B_\eta(\eta_0, \psi) = \sum_{n=1}^{\infty} (B_n(\eta_0) \sin n\psi + A_n(\eta_0) \cos n\psi)$.

However, the coefficients in the two series expressions cannot be identified as in the case of the circular harmonics, because the metric coefficient h_2 is a function of ψ. To overcome this problem we define the field components $\tilde{B}_\eta$ and $\tilde{B}_\psi$ by the coordinate derivative without metric coefficients $\tilde{B}_\eta = \frac{\partial A_z}{\partial \psi}$, and $\tilde{B}_\psi = -\frac{\partial A_z}{\partial \eta}$. This corresponds to taking the exterior derivative in differential-form calculus and results in covariant components of the magnetic flux density. $\tilde{B}_\eta$ can then be expressed as $\tilde{B}_\eta(\eta, \psi) = \sum_{n=1}^{\infty} (n\mathcal{A}_n \sinh n\eta \cos n\psi - n\mathcal{B}_n \cosh n\eta \sin n\psi)$.

At the reference ellipse, $\eta = \eta_0$, the field component $\tilde{B}_\eta$ can now be calculated from the Cartesian components by $\tilde{B}_\eta = e \sinh \eta \cos \psi B_x + e \cosh \eta \sin \psi B_y$ and expressed as the Fourier series $\tilde{B}_\eta(\eta_0, \psi) = \sum_{n=1}^{\infty} \left(\tilde{B}_n(\eta_0) \sin n\psi + \tilde{A}_n(\eta_0) \cos n\psi \right)$, where $\tilde{A}_n(\eta_0)$ and $\tilde{B}_n(\eta_0)$ are the ordinary Fourier coefficients of the $\tilde{B}_\eta$ field component. Comparing the coefficients in the two series expressions yields $\mathcal{A}_n = (n \sinh n\eta_0)^{-1} \tilde{A}_n(\eta_0)$, and $\mathcal{B}_n = -(n \cosh n\eta_0)^{-1} \tilde{B}_n(\eta_0)$. To avoid the calculation of the flux density by numerical differentiation, it is again possible to perform a Fourier series expansion of the vector potential at the reference ellipse, i.e., the magnetic flux density is directly expressed as a function of the multipoles obtained from the series expansion of the vector potential at η_0.

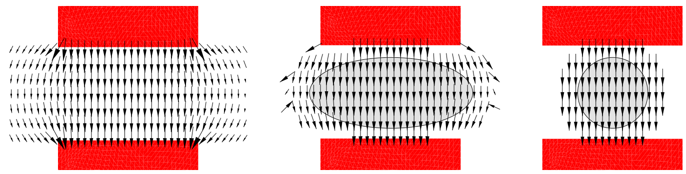

Fig. 1. Left: Numerically calculated field distribution between the poles of a dipole magnet. Middle and right: Field calculated from the truncated series. Middle: Elliptic coordinates (n = 40, a = 70 mm, b = 30 mm). Right: Circular coordinates (n = 40, r_0 = 30 mm).

Fig. 1 (left) shows the numerically calculated field between the poles of a dipole magnet. The field resulting from the truncated series is shown in the middle. Notice how numerical errors dominate the field solution outside of the reference ellipse; all field vectors that deviate by more than 20% in amplitude are omitted from the plot. The advantage of the elliptic multipole expansion is obvious from the comparison with the truncated series of the circular multipole expansion (right). The effect of the fringe field is better modeled in the elliptic coordinate system and thus the elliptic multipole coefficients are better suited to the optimization of dipole magnets with a large aspect ratio of their air gaps.

We have thus found a convenient way to describe the field imperfections in accelerator magnets. We have been able to reconstitute the field in the entire aperture of the magnet from measurements on its circular or elliptic boundary and derived the scaling laws to limit the field evaluation to the boundary of the problem domain and use only the vector potential, without the need for numerical differentiation.

Moreover, this technique allows us to measure the multipoles by means of the oscillating-wire method [2], whereby a wire is stretched between two precision stages allowing to position the wire step-by-step on $k = 0, \cdots, K-1$ generators of a cylindrical or elliptical domain encompassing the magnet aperture. When the wire is fed by a sinusoidal current, the resulting oscillation amplitudes will become a 2π-periodic function of the position, which can then be treated with the discrete Fourier transform.

REFERENCES

[1] S. Russenschuck, S., Field Computation for Accelerator Magnets, Wiley-VCH, 2010

[2] P. Arpaia, M. Buzio. J. Garcia Perez, C. Petrone, S. Russenschuck, L. Walckiers, "Measuring field multipoles in accelerator magnets with small-apertures by an oscillating wire moved on a circular trajectory," *IOP Journal of Instrumentation*, 2012.

[3] P. Schnizer, B. Schnizer, P. Akishin, E. Fischer, "Theory and application of plane elliptic multipoles for static magnetic fields," *Nuclear Instruments and Methods in Physics Research A*, Elsevier, 2009